

# Synchronizing automata with random inputs

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**Abstract.** We study the problem of synchronization of automata with random inputs. We present a series of automata such that the expected number of steps until synchronization is exponential in the number of states. At the same time, we show that the expected number of letters to synchronize any pair of the famous Černý automata is at most cubic in the number of states.

## 1 Introduction

A *complete deterministic finite automaton*  $\mathcal{A}$ , or simply *automaton*, is a triple  $\langle Q, \Sigma, \delta \rangle$ , where  $Q$  is a finite *set of states*,  $\Sigma$  is a finite *input alphabet*, and  $\delta : Q \times \Sigma \mapsto Q$  is a totally defined *transition function*. Following standard notation, by  $\Sigma^*$  we mean the set of all finite words over the alphabet  $\Sigma$ , including the empty word  $\varepsilon$ . The function  $\delta$  naturally extends to the free monoid  $\Sigma^*$ ; this extension is still denoted by  $\delta$ . Thus, via  $\delta$ , every word  $w \in \Sigma^*$  acts on the set  $Q$ .

An automaton  $\mathcal{A}$  is called *synchronizing*, if there is a word  $w \in \Sigma^*$  which brings all states of the automaton  $\mathcal{A}$  to a particular one, i.e. there exists a state  $t \in Q$  such that  $\delta(s, w) = t$  for every  $s \in Q$ . Any such word  $w$  is said to be a *reset* (or *synchronizing*) *word* for the automaton  $\mathcal{A}$ . The minimum length of reset words for  $\mathcal{A}$  is called the *reset threshold* of  $\mathcal{A}$ . Note, that the language  $\mathcal{L}$  of synchronizing words of the automaton  $\mathcal{A}$  is a *two-sided ideal*, i.e.  $\Sigma^* \mathcal{L} \Sigma^* = \mathcal{L}$ . We say that the word  $w$  synchronizes a pair  $\{s, t\}$  if  $\delta(s, w) = \delta(t, w)$ .

Synchronizing automata serve as transparent and natural models of error-resistant systems in many applied areas such as robotics, coding theory, and bioinformatics. At the same time, synchronizing automata surprisingly arise in some parts of pure mathematics: algebra, symbolic dynamics, and combinatorics on words. See recent surveys by Sandberg [8] and Volkov [11] for a general introduction to the theory of synchronizing automata.

The interest to the field is heated also by the famous *Černý conjecture*. In 1964 Černý exhibited a series  $\mathcal{C}_n$  of automata with  $n$  states whose

reset threshold equals  $(n - 1)^2$  [3]. Soon after he conjectured, that this series represents the worst possible case, i.e. the reset threshold of every  $n$ -state synchronizing automaton is at most  $(n - 1)^2$ . In spite of its simple formulation and intensive researchers' efforts, the Černý conjecture remains unresolved for fifty years. The best known upper bound on the reset threshold of a synchronizing  $n$ -state automaton is  $\frac{n^3 - n}{6}$  by Pin [6].

The focus of this paper is on probabilistic aspects of synchronization. One general question that was actively studied in the literature is the following: what are synchronizing properties of a *random automaton*? Skvortsov and Zaks have shown that a random automaton with sufficiently large number of letters is synchronizing with high probability [9]. Later on, they proved that a random 4-letter automaton is synchronizing with a positive probability that is independent of the number of states [13]. The last step in this direction seems to be done by Berlinkov [2]. He has shown that a random automaton over a binary alphabet is synchronizing with high probability. Another direction within this setting is devoted to reset thresholds of random synchronizing automata. It was shown in [9] that a random automaton with large number of letters satisfies the Černý conjecture with high probability. Furthermore, computational experiments performed in [10,5] suggest that expected reset threshold of a random synchronizing automaton is sub-linear.

The setting of the present paper is different. In our considerations we investigate how *random input* acts on a *fixed* automaton. Assume that several copies of a synchronizing automaton  $\mathcal{A}$  simultaneously read a common input from a fixed source of random letters. Initially these automata may be in different states. What is the expected number steps  $E$  until all copies will be in the same state? We can give the following illustration of this approach. Let  $\mathcal{D}$  be a decoder of a code. Due to data transmission errors the decoder  $\mathcal{D}$  may be in a different state compared to a correct decoder  $\mathcal{D}_c$ . Then the number  $E$  computed for decoders  $\mathcal{D}$  and  $\mathcal{D}_c$  represents an average number of steps before recovery of the decoder  $\mathcal{D}$  after an error.

Our setting heavily depends on a model of a random input. In the present paper we restrict ourselves with a binary alphabet  $\Sigma = \{a, b\}$  and the *Bernoulli model*, i.e. every succeeding letter is drawn independently with probability  $p$  for the letter  $a$  and probability  $q = 1 - p$  for the letter  $b$ . In section 2 we present a series of  $n$ -state automata  $\mathcal{U}_n$  over  $\Sigma$  and a pair  $S$  such that the expected number of steps to synchronize  $S$  is exponential in  $n$ . At the same time, in section 3 we show that the

expected number of steps to synchronize any pair of the famous example  $\mathcal{C}_n$  by Černý is at most cubic in  $n$ . These results reveal that despite the fact that synchronization of  $\mathcal{C}_n$  is hard in the deterministic case, it is relatively easy in the random setting.

## 2 Automata $\mathcal{U}_n$ with the sink state

Let  $\Sigma$  be a binary alphabet  $\{a, b\}$ . Let  $\mathcal{U}_n$  be the minimal automaton recognizing the language  $L_n$ , where  $L_n$  is equal to  $\Sigma^* a^{\frac{n+1}{2}} b^{\frac{n-1}{2}} \Sigma^*$  if  $n$  is odd, and to  $\Sigma^* a^{\frac{n}{2}} b^{\frac{n}{2}} \Sigma^*$  if  $n$  is even. Note, that the automaton  $\mathcal{U}_n$  is synchronizing, and its language of synchronizing words coincides with  $L_n$ .

First, we will consider the case when  $n$  is odd. Let us define  $\mathcal{U}_n$  more formally, see fig. 1. The set of states of  $\mathcal{U}_n$  is equal to  $\{1, 2, \dots, n+1\}$ . The transition function  $\delta$  of  $\mathcal{U}_n$  is defined as follows:

$$\delta(i, a) = \begin{cases} i+1, & \text{if } i < \frac{n+3}{2} \\ i, & \text{if } i = \frac{n+3}{2} \\ 2, & \text{if } \frac{n+3}{2} < i < n+1 \\ n+1, & \text{if } i = n+1; \end{cases} \quad \delta(i, b) = \begin{cases} 1, & \text{if } i < \frac{n+3}{2} \\ i+1, & \text{if } \frac{n+3}{2} \leq i < n+1 \\ n+1, & \text{if } i = n+1. \end{cases}$$

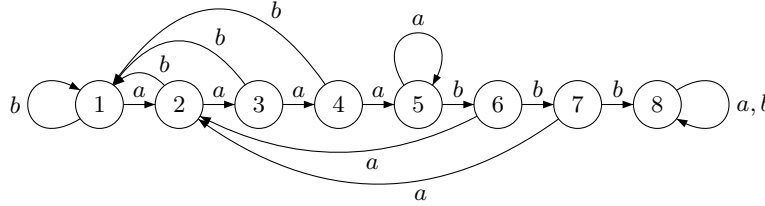


Fig. 1. Automaton  $\mathcal{U}_7$

Let  $\mathcal{B}(p, q)$  be the source of random letters such that each letter is drawn independently with probability  $p$  for the letter  $a$  and probability  $q = 1-p$  for the letter  $b$ . Let  $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$  be a synchronizing automaton. We consider the following random process:

1.  $S := Q$
2. **Until**  $|S| = 1$  **do**
3.      $x \leftarrow \mathcal{B}(p, q)$
4.      $S := \delta(S, x)$

We start with the set  $S$  equal to the state set  $Q$ . On each step we draw a

random letter  $x$  from the source  $\mathcal{B}(p, q)$  and apply it to  $S$ . We stop when  $S$  is a singleton.

In general, we are interested in the average number of steps that this process takes for a given automaton  $\mathcal{A}$ . In particular, we have the following theorem.

**Theorem 1.** *Let  $n$  be a positive odd integer. The expected number of letters, that are drawn from  $\mathcal{B}(p, q)$ , until  $\mathcal{U}_n$  is synchronized, is equal to  $\frac{1}{p^{\frac{n+1}{2}} q^{\frac{n-1}{2}}}$ .*

*Proof.* It is rather easy to see that the word  $w$  synchronizes the automaton  $\mathcal{U}_n$  if and only if  $\delta(1, w) = n + 1$ . Thus, the average number of steps in our random process equals the average length of a random walk that brings the state 1 to the state  $n + 1$ , where the probability of the transition labeled by  $a$  is  $p$ , and the probability of the transition labeled by  $b$  is  $q$ . It is well-known how to compute the latter quantity<sup>1</sup> [7, section 6.2]. For  $1 \leq i \leq n + 1$  let  $\mu_i$  be the expected length of a random walk that brings the state  $i$  to the state  $n + 1$ . These quantities necessarily satisfy the following system of equations:

$$\begin{cases} \mu_1 = p\mu_2 + q\mu_1 + 1 & (1) \\ \mu_i = p\mu_{i+1} + q\mu_1 + 1, \text{ if } 1 \leq i \leq \frac{n+1}{2} & (2) \\ \mu_{\frac{n+3}{2}} = p\mu_{\frac{n+3}{2}} + q\mu_{\frac{n+5}{2}} + 1 & (3) \\ \mu_i = p\mu_2 + q\mu_{i+1} + 1, \text{ if } \frac{n+5}{2} \leq i \leq n - 1 & (4) \\ \mu_n = p\mu_2 + q\mu_{n+1} + 1 & (5) \\ \mu_{n+1} = 0 & (6) \end{cases}$$

We will solve this system in several steps:

1. Let us show that  $\mu_i = \mu_1 - \frac{p^{i-1}-1}{p^i-p^{i-1}}$  for  $2 \leq i \leq \frac{n+3}{2}$ . Equation (1) implies that this statement is true for  $i = 2$ . Suppose now that the statement is true for  $\mu_i$ . Let us show that it is true for  $\mu_{i+1}$ . From equation (2) we get  $\mu_1 - \frac{p^{i-1}-1}{p^i-p^{i-1}} = p\mu_{i+1} + q\mu_1 + 1$ . Therefore,  $\mu_1 - \frac{p^i-1}{p^{i+1}-p^i} = \mu_{i+1}$ .

We will denote  $\frac{p^{\frac{n+1}{2}}-1}{p^{\frac{n+3}{2}}-p^{\frac{n+1}{2}}}$  as  $C$  in order to simplify notation. Therefore,  $\mu_{\frac{n+3}{2}} = \mu_1 - C$ .

2. Equation (3) immediately implies  $\mu_{\frac{n+5}{2}} = \mu_{\frac{n+3}{2}} - \frac{1}{q}$ . Therefore, we have  $\mu_{\frac{n+5}{2}} = \mu_1 - C - \frac{1}{q}$ .

3. Now we will show that  $\mu_i = \mu_1 - \frac{C}{q^{i-\frac{n+3}{2}}} - \frac{1}{q^{i-\frac{n+3}{2}}}$  for  $\frac{n+5}{2} \leq i \leq n$ .

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<sup>1</sup> It is also called the mean absorption time of a Markov chain

This statement is true for  $i = \frac{n+5}{2}$ . Let us show that it is true for every succeeding  $i \leq n$ . Since  $\mu_2 \stackrel{(1)}{=} \mu_1 - \frac{1}{p}$  we can rewrite equation (4) in the following way:  $\mu_i = p\mu_1 + q\mu_{i+1}$ . Our assumption states that  $\mu_i = \mu_1 - \frac{C}{q^{i-\frac{n+5}{2}}} - \frac{1}{q^{i-\frac{n+3}{2}}}$ . Therefore,  $q\mu_1 - \frac{C}{q^{i-\frac{n+5}{2}}} - \frac{1}{q^{i-\frac{n+3}{2}}} = q\mu_{i+1}$ . Finally,  $\mu_{i+1} = \mu_1 - \frac{C}{q^{i-\frac{n+3}{2}}} - \frac{1}{q^{i-\frac{n+1}{2}}}$ .

Note, we have  $\mu_n = \mu_1 - \frac{C}{q^{\frac{n-5}{2}}} - \frac{1}{q^{\frac{n-3}{2}}}$ .

4. Equation (5) and (6) imply  $\mu_n = p\mu_1$ .

Therefore,  $\mu_1 = \frac{C}{q^{\frac{n-3}{2}}} + \frac{1}{q^{\frac{n-1}{2}}} = \frac{qC+1}{q^{\frac{n-1}{2}}} = \frac{1}{p^{\frac{n+1}{2}} q^{\frac{n-1}{2}}}$ .

Slightly modifying the argument of the previous theorem we can obtain a similar result, when  $n$  is even.

**Theorem 2.** *Let  $n$  be a positive even integer. The expected number of letters, that are drawn from  $\mathcal{B}(p, q)$ , until  $\mathcal{U}_n$  is synchronized, is equal to  $\frac{1}{p^{\frac{n}{2}} q^{\frac{n}{2}}}$ .*

### 3 The Černý automata $\mathcal{C}_n$

Now we study a classical example introduced by Černý in 1964 [3]. Recall the definition of the Černý automaton  $\mathcal{C}_n$ , see fig. 2. The state set of  $\mathcal{C}_n$  is  $Q = \{0, 1, \dots, n-1\}$ , and the letters  $a$  and  $b$  act on  $Q$  as follows.

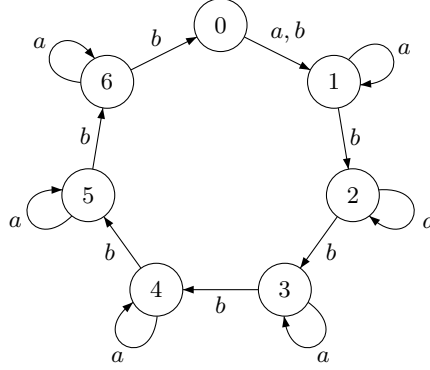
$$\delta(i, a) = \begin{cases} 1 & \text{if } i = 0, \\ i & \text{if } i > 0; \end{cases} \quad \delta(i, b) = \begin{cases} i+1 & \text{if } i < n-1, \\ 0 & \text{if } i = n-1. \end{cases}$$

The reset threshold of  $\mathcal{C}_n$  is equal to  $(n-1)^2$ , see [1,4,3].

The goal of the present section is to find the expected number of letters, that are drawn from  $\mathcal{B}(p, q)$ , until the pair of states  $\{1, \frac{n+1}{2}\}$ , when  $n$  is odd, and the pair  $\{1, \frac{n+2}{2}\}$ , when  $n$  is even, is synchronized. At the same time, we will see that the expectation for these pairs is the largest among other pairs.

Let  $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$  be an automaton. The *pair automaton*  $\mathcal{P}(\mathcal{A})$  is defined as follows. The set of states of  $\mathcal{P}(\mathcal{A})$  is equal to  $\{\{s, t\} \mid s \neq t\} \cup \{\mathbf{z}\}$ . The transition function  $\delta_{\mathcal{P}}$  of  $\mathcal{P}(\mathcal{A})$  for each  $x \in \Sigma$ ,  $s, t \in Q$  is defined by the following rules:

$$\delta_{\mathcal{P}}(\{s, t\}, x) = \begin{cases} \{\delta(s, x), \delta(t, x)\}, & \text{if } \delta(s, x) \neq \delta(t, x) \\ \mathbf{z}, & \text{if } \delta(s, x) = \delta(t, x); \end{cases} \quad \delta_{\mathcal{P}}(\mathbf{z}, x) = \mathbf{z}.$$



**Fig. 2.** The automaton  $\mathcal{C}_7$

Note, that all words  $w$  that synchronize a pair  $\{s, t\}$  label a path in  $\mathcal{P}(\mathcal{A})$  from  $\{s, t\}$  to  $\mathbf{z}$ . Furthermore, a word  $w$  is synchronizing for  $\mathcal{A}$  if and only if  $w$  is synchronizing for  $\mathcal{P}(\mathcal{A})$ . The proof of this easy fact can be found for instance in [11].

First, let  $n$  be a positive odd integer. In order to prove the main result of this section we will require another representation of the pair automaton of  $\mathcal{C}_n$ . We will denote it by  $\mathcal{P}_n$ , see fig. 3. The state set of  $\mathcal{P}_n$  is the set of ordered pairs

$$\{(i, \ell) \mid 0 \leq i \leq n-1, 1 \leq \ell \leq \frac{n-1}{2}\} \cup \{\mathbf{z}\}.$$

The transition function  $\delta$  is defined as follows.

$$\delta(\mathbf{z}, x) = \mathbf{z} \text{ for every } x \in \Sigma,$$

$$\delta((i, \ell), b) = ((i+1) \bmod n, \ell) \text{ for every admissible } i \text{ and } \ell.$$

$\delta((i, \ell), a) = (i, \ell)$  for every admissible  $i$  and  $\ell$  with the exception of the following cases:

$$\delta((0, 1), a) = \mathbf{z},$$

$$\delta((0, \ell), a) = (1, \ell-1) \text{ if } 2 \leq \ell \leq \frac{n-1}{2},$$

$$\delta((n-\ell, \ell), a) = (n-\ell, \ell+1) \text{ if } 1 \leq \ell \leq \frac{n-3}{2},$$

$$\delta((\frac{n+1}{2}, \frac{n-1}{2}), a) = (1, \frac{n-1}{2}).$$

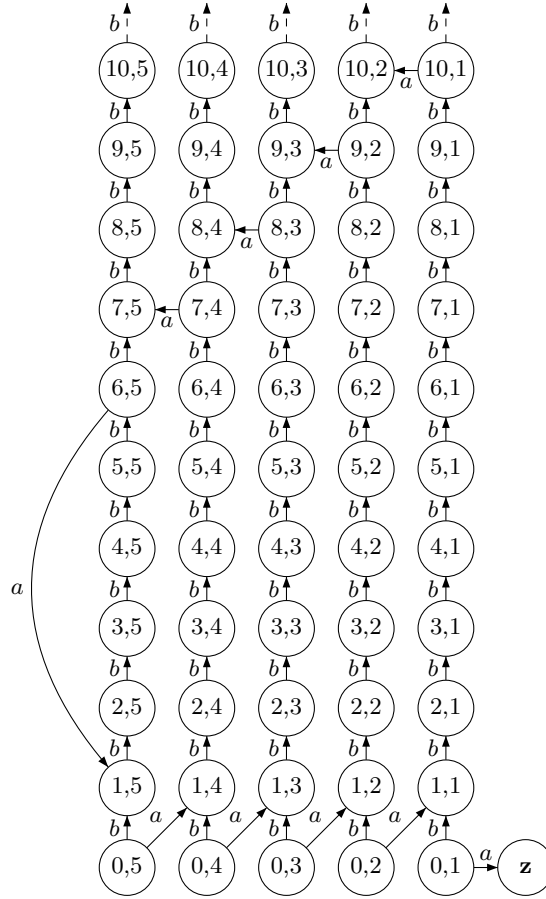
**Lemma 1.** *Let  $n$  be a positive odd integer. The automaton  $\mathcal{P}_n$  is isomorphic to the pair automaton of  $\mathcal{C}_n$ .*

*Proof.* We will construct the desired isomorphism. The sink state  $\mathbf{z}$  of the pair automaton is mapped to the sink state  $\mathbf{z}$  of  $\mathcal{P}_n$ . Let  $\{s, t\}$  be an arbitrary pair of states. Let  $\delta_{\mathcal{C}}$  be the transition function of the automaton  $\mathcal{C}_n$ . There is a positive integer  $m$  that satisfies equations  $\delta_{\mathcal{C}}(s, b^m) = t$

and  $\delta_{\mathcal{C}}(t, b^{n-m}) = s$ . Let  $\ell$  be the minimum of  $m$  and  $n - m$ . Since  $n$  is odd  $m \neq n - m$ . Let

$$i = \begin{cases} s, & \text{if } \delta_{\mathcal{C}}(s, b^{\ell}) = t \\ t, & \text{if } \delta_{\mathcal{C}}(t, b^{\ell}) = s. \end{cases}$$

Then the pair  $\{s, t\}$  of the pair automaton is mapped to the state  $(i, \ell)$  of the automaton  $\mathcal{P}_n$ . It is easy to check that the presented mapping is an isomorphism.



**Fig. 3.** Pair automaton of  $\mathcal{C}_{11}$

Now we are ready to formulate the main result of this section.

**Theorem 3.** Let  $n$  be a positive odd integer. The expected number of letters, that are drawn from  $\mathcal{B}(p, q)$ , until the pair  $\{1, \frac{n+1}{2}\}$  of  $\mathcal{C}_n$  is synchronized, is equal to  $\frac{(n-1)((n-1)^2 + q(3n-5) + 4q^2)}{8pq^2}$ .

*Proof.* It is not hard to see that a word  $w$  labels a path from  $(i, \ell)$  to  $\mathbf{z}$  in the automaton  $\mathcal{P}_n$  if and only if the word  $w$  synchronizes the pair  $\{i, (i + \ell) \bmod n\}$  of the automaton  $\mathcal{C}_n$ . Thus, the expected number of letters until the pair  $\{1, \frac{n+1}{2}\}$  is synchronized is equal to the expected length of a random walk in automaton  $\mathcal{P}_n$  from the state  $(1, \frac{n-1}{2})$  to the state  $\mathbf{z}$ , where the probability of the transition labeled by  $a$  is  $p$ , and the probability of the transition labeled by  $b$  is  $q$ . For  $0 \leq i \leq n-1$  and  $1 \leq \ell \leq \frac{n-1}{2}$  let  $\mu_{i,\ell}$  be the expected length of a random walk that brings the state  $(i, \ell)$  of  $\mathcal{P}_n$  to the state  $\mathbf{z}$ . As in the proof of the theorem 1 these values have to satisfy a particular system of linear equations, see [7, section 6.2]. For convenience, we will split this system into three parts. The first part:

$$\begin{cases} \mu_{0,1} = q\mu_{1,1} + 1 & (1) \\ \mu_{i,1} = p\mu_{i,1} + q\mu_{i+1,1} + 1, & \text{if } 1 \leq i \leq n-2 & (2) \\ \mu_{n-1,1} = p\mu_{n-1,2} + q\mu_{0,1} + 1 & (3) \\ \mu_{\mathbf{z}} = 0 \end{cases}$$

The second part,  $2 \leq \ell \leq \frac{n-3}{2}$ :

$$\begin{cases} \mu_{0,\ell} = p\mu_{1,\ell-1} + q\mu_{1,\ell} + 1 & (4) \\ \mu_{i,\ell} = p\mu_{i,\ell} + q\mu_{i+1,\ell} + 1, & \text{if } 1 \leq i \leq n-\ell-1 & (5) \\ \mu_{n-\ell,\ell} = p\mu_{n-\ell,\ell+1} + q\mu_{n-\ell+1,\ell} + 1, & (6) \\ \mu_{i,\ell} = p\mu_{i,\ell} + q\mu_{i+1,\ell} + 1, & \text{if } n-\ell+1 \leq i \leq n-2 & (7) \\ \mu_{n-1,\ell} = p\mu_{n-1,\ell} + q\mu_{0,\ell} + 1, & (8) \end{cases}$$

And the third part:

$$\begin{cases} \mu_{0, \frac{n-1}{2}} = p\mu_{1, \frac{n-3}{2}} + q\mu_{1, \frac{n-1}{2}} + 1 & (9) \\ \mu_{i, \frac{n-1}{2}} = p\mu_{i, \frac{n-1}{2}} + q\mu_{i+1, \frac{n-1}{2}} + 1, & \text{if } 1 \leq i \leq \frac{n-1}{2} & (10) \\ \mu_{\frac{n+1}{2}, \frac{n-1}{2}} = p\mu_{1, \frac{n-1}{2}} + q\mu_{\frac{n+3}{2}, \frac{n-1}{2}} + 1, & (11) \\ \mu_{i, \frac{n-1}{2}} = p\mu_{i, \frac{n-1}{2}} + q\mu_{i+1, \frac{n-1}{2}} + 1, & \text{if } \frac{n+3}{2} \leq i \leq n-2 & (12) \\ \mu_{n-1, \frac{n-1}{2}} = p\mu_{n-1, \frac{n-1}{2}} + q\mu_{0, \frac{n-1}{2}} + 1, & (13) \end{cases}$$

Let us resolve the first part. Applying equations (2) in successive order we get  $\mu_{1,1} \stackrel{(2)}{=} \mu_{n-1,1} + \frac{n-2}{q} \stackrel{(3)}{=} p\mu_{n-1,2} + q\mu_{0,1} + 1 + \frac{n-2}{q}$ . Since



$\mu_{n-1,2} \stackrel{(8)}{=} \mu_{0,2} + \frac{1}{q} \stackrel{(4)}{=} p\mu_{1,1} + q\mu_{1,2} + 1 + \frac{1}{q}$  and  $\mu_{0,1} \stackrel{(1)}{=} q\mu_{1,1} + 1$  we have  $\mu_{1,1} = p(p\mu_{1,1} + q\mu_{1,2} + 1 + \frac{1}{q}) + q(q\mu_{1,1} + 1) + 1 + \frac{n-2}{q}$ . After trivial simplification, using the fact that  $1 - p^2 - q^2 = 2pq$ , we obtain

$$2\mu_{1,1} = \mu_{1,2} + \frac{n-p}{pq^2} \quad (14)$$

Let us focus on the second part. Let  $2 \leq \ell \leq \frac{n-3}{2}$ . Applying equations (5) several times in successive order we get  $\mu_{1,\ell} \stackrel{(5)}{=} \mu_{n-\ell,\ell} + \frac{n-\ell-1}{q} \stackrel{(6)}{=} p\mu_{n-\ell,\ell+1} + q\mu_{n-\ell+1,\ell} + 1 + \frac{n-\ell-1}{q}$ . Since  $\mu_{n-\ell,\ell+1} \stackrel{(7 \text{ or } 12)}{=} \mu_{n-1,\ell+1} + \frac{\ell-1}{q} \stackrel{(8 \text{ or } 13)}{=} \mu_{0,\ell+1} + \frac{\ell}{q} \stackrel{(4 \text{ or } 9)}{=} p\mu_{1,\ell} + q\mu_{1,\ell+1} + 1 + \frac{\ell}{q}$  and  $\mu_{n-\ell+1,\ell} \stackrel{(7)}{=} \mu_{n-1,\ell} + \frac{\ell-2}{q} \stackrel{(8)}{=} \mu_{0,\ell} + \frac{\ell-1}{q} \stackrel{(4)}{=} p\mu_{1,\ell-1} + q\mu_{1,\ell} + 1 + \frac{\ell-1}{q}$  we have  $\mu_{1,\ell} = p(p\mu_{1,\ell} + q\mu_{1,\ell+1} + 1 + \frac{\ell}{q}) + q(p\mu_{1,\ell-1} + q\mu_{1,\ell} + 1 + \frac{\ell-1}{q}) + 1 + \frac{n-\ell-1}{q}$ . After simplification we obtain the following equation:

$$2\mu_{1,\ell} = \mu_{1,\ell+1} + \mu_{1,\ell-1} + \frac{n-p}{pq^2} \quad (15)$$

Let us resolve the third part. Applying equations (10) in successive order we get  $\mu_{1,\frac{n-1}{2}} \stackrel{(10)}{=} \mu_{\frac{n+1}{2},\frac{n-1}{2}} + \frac{n-1}{2q} \stackrel{(11)}{=} p\mu_{1,\frac{n-1}{2}} + q\mu_{\frac{n+3}{2},\frac{n-1}{2}} + 1 + \frac{n-1}{2q}$ . Since  $\mu_{\frac{n+3}{2},\frac{n-1}{2}} \stackrel{(12)}{=} \mu_{n-1,\frac{n-1}{2}} + \frac{n-5}{2q} \stackrel{(13)}{=} \mu_{0,\frac{n-1}{2}} + \frac{n-3}{2q} \stackrel{(9)}{=} p\mu_{1,\frac{n-3}{2}} + q\mu_{1,\frac{n-1}{2}} + 1 + \frac{n-3}{2q}$  we have  $\mu_{1,\frac{n-1}{2}} = p\mu_{1,\frac{n-1}{2}} + q(p\mu_{1,\frac{n-3}{2}} + q\mu_{1,\frac{n-1}{2}} + 1 + \frac{n-3}{2q}) + 1 + \frac{n-1}{2q}$ . After an easy simplification we obtain the following equation:

$$\mu_{1,\frac{n-1}{2}} = \mu_{1,\frac{n-3}{2}} + \frac{q^2 + \frac{n-1}{2}q + \frac{n-1}{2}}{pq^2} \quad (16)$$

Summing up equations (14),(15) for  $2 \leq \ell \leq \frac{n-3}{2}$ , and (16) we obtain the following equation:

$$\mu_{1,1} = \frac{n-3}{2} \cdot \frac{n-p}{pq^2} + \frac{q^2 + \frac{n-1}{2}q + \frac{n-1}{2}}{pq^2} \quad (17)$$

Now we can show that

$$\mu_{1,\ell} = \ell\mu_{1,1} - \frac{\ell(\ell-1)}{2} \cdot \frac{n-p}{pq^2} \quad (18)$$

Equation (14) serves as the induction base. Using equation (15) we make the induction step.

From equation (18) for  $\ell = \frac{n-1}{2}$  we get  $\mu_{1, \frac{n-1}{2}} = \frac{n-1}{2}\mu_{1,1} - \frac{(n-1)(n-3)}{8} \cdot \frac{n-p}{pq^2}$ . Using (17) after tedious simplification we get the final result:

$$\mu_{1, \frac{n-1}{2}} = \frac{(n-1)((n-1)^2 + q(3n-5) + 4q^2)}{8pq^2} \quad (19)$$

Note, that the leading term of  $\mu_{1, \frac{n-1}{2}}$  is equal to  $\frac{n^3}{8pq^2}$ . It is easy to see, that the minimum of  $\frac{1}{8pq^2}$  is reached at  $p = \frac{1}{3}$ . Therefore, the expected number of letters until the pair  $\{1, \frac{n+1}{2}\}$  of  $\mathcal{C}_n$  is synchronized is close to the minimum for the source of random letters  $\mathcal{B}(\frac{1}{3}, \frac{2}{3})$ . In this case we have

$$\mu_{1, \frac{n-1}{2}} = \frac{27n^3}{32} - \frac{27n^2}{32} - \frac{15n}{32} + \frac{15}{32}$$

**Theorem 4.** *Let  $n$  be a positive even integer. The expected number of letters, that are drawn from  $\mathcal{B}(p, q)$ , until the pair  $\{1, \frac{n+2}{2}\}$  of  $\mathcal{C}_n$  is synchronized, is equal to  $\frac{n((n-1)(n-2) + q(3n-6) + 4q^2)}{8pq^2}$ .*

*Proof.* The proof this theorem is similar to a proof of a previous one and we will omit it. The main difference lies in the equations (9) – (13). So, instead of (16) we will get

$$\mu_{1, \frac{n}{2}} = \mu_{1, \frac{n-2}{2}} + \frac{\frac{n-2}{2} + q}{pq} \quad (20)$$

From (18) and (20) we can obtain the following equation:

$$\mu_{1, \frac{n}{2}} = \frac{n((n-1)(n-2) + q(3n-6) + 4q^2)}{8pq^2}.$$

As before, the expected number of letters until the pair  $\{1, \frac{n+2}{2}\}$  of  $\mathcal{C}_n$  is synchronized is close to the minimum for the the source  $\mathcal{B}(\frac{1}{3}, \frac{2}{3})$ :

$$\frac{27n^3}{32} - \frac{27n^2}{32} - \frac{6n}{32}$$

## 4 Conclusion

The expected number of steps until synchronization of the automata  $\mathcal{U}_n$  is exponential in the number of states. At the same time, the expected number of steps to synchronize any pair of the Černý automata  $\mathcal{C}_n$  is at most cubic in the number of states. These results reveal that despite the fact that synchronization of  $\mathcal{C}_n$  is hard in the deterministic case, it is relatively easy in the random setting.

## References

1. Ananichev, D. S., Gusev, V. V., Volkov M. V.: Primitive digraphs with large exponents and slowly synchronizing automata. *Journal of Mathematical Sciences (US)*, 192(3), 263–278 (2013)
2. Berlinkov, M. V.: On the probability of being synchronizable. <http://arxiv.org/abs/1304.5774> (2013)
3. Černý, J.: Poznámka k homogénnym eksperimentom s konečnými automatami. *Matematicko-fyzikálny Časopis Slovensk. Akad. Vied* 14(3) 208–216 (1964) (in Slovak)
4. Gusev, V. V.: Lower bounds for the length of reset words in eulerian automata. *Int. J. Found. Comput. Sci.*, 24(2), 251–262 (2013)
5. Kisielewicz, A., Kowalski J., Szykuła, M.: A Fast Algorithm Finding the Shortest Reset Words. *Lect. Notes Comput. Sci.* 7936, 182–196 (2013)
6. Pin, J.-E.: On two combinatorial problems arising from automata theory. *Ann. Discrete Math.* 17, 535–548 (1983)
7. Privault, N.: *Understanding Markov Chains*. Springer (2013)
8. Sandberg, S.: Homing and synchronizing sequences. *Model-Based Testing of Reactive Systems. Lect. Notes Comput. Sci.* 3472, 5–33 (2005)
9. Skvortsov, E. S., Zaks, Yu.: Synchronizing random automata. *Discr. Math. and Theor. Comp. Sci.* 12(4), 95–108 (2010)
10. Skvortsov, E., Tipikin, E.: Experimental study of the shortest reset word of random automata. *CIAA 2011, Lect. Notes Comput. Sci.*, 6807, 290–298 (2011)
11. Volkov, M. V.: Synchronizing automata and the Černý conjecture. *Languages and Automata: Theory and Applications, Lect. Notes Comput. Sci.* 5196, 11–27 (2008)
12. Volkov, M. V.: Synchronizing automata preserving a chain of partial orders. *Theoret. Comput. Sci.* 410, 2992–2998 (2009)
13. Zaks, Yu. I., Skvortsov, E. S.: Synchronizing random automata on a 4-letter alphabet. *Journal of Mathematical Sciences*, 192(3), 303–306 (2013)